

Phase synchronization in coupled nonidentical excitable systems and array-enhanced coherence resonance

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We study the dynamics of a lattice of coupled nonidentical Fitz Hugh-Nagumo system subject to independent external noise. It is shown that these stochastic oscillators can lead to global synchronization behavior *without an external signal*. With the increase of the noise intensity, the system exhibits coherence resonance behavior. Coupling can enhance greatly the noise-induced coherence in the system.

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The study of coupled oscillators is one of the fundamental problems with applications in various fields [1]. Mutual synchronization of the oscillators is of great interest and importance among the collective dynamics of the coupled oscillators. The notation of synchronization has been extended to include a variety of phenomena in the context of interacting chaotic oscillators, such as complete synchronization [2], generalized synchronization [3], phase synchronization [4,5] and lag synchronization [6]. Also, synchronization phenomenon has been studied in stochastic systems. Stochastic resonance can be understood from the viewpoint of frequency locking and phase synchronization of the noise-induced motion to the external signal [7]. Due to this synchronization, stochastic bistable elements subject to the *same* periodic signal, when appropriately coupled, can display global synchronization to the external periodic signal. This synchronization has the effect of enhancing the stochastic resonance in the array [8].

Noise-induced coherent motion has been a topic of great interest recently. For a system at a saddle-node bifurcation point, an optimal amount of noise can induce the most coherent motion in the system [9]. More generally, noise-induced coherent motion has been demonstrated in a variety of excitable systems [10–12] and systems with delay [13]. Although described by different terms such as stochastic resonance without external periodic force, autonomous stochastic resonance or coherence resonance, a common feature of this type of system is the increased coherence of the motion and resonantlike behavior of the coherence induced *purely by noise without an external signal*. An interesting question about this type of system is how these stochastic elements behave when coupled. Do the elements that are subjected to independent noise display synchronization? Can the synchronization enhance the coherence in the system motion? A recent study demonstrated noise-induced global oscillation and resonant behavior in a subexcitable media [14]. In Ref. [15], it is shown that *two* interacting coherence oscillators can be synchronized. However, whether the coherence of the motion has been enhanced by the interaction between the elements is not clear.

We address these problems in this Rapid Communication by the following N coupled *nonidentical* Fitz Hugh-Nagumo (FHN) neurons, a simple but representative model of excitable system and nerve pulses [10],

$$\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + g(x_{i+1} + x_{i-1} - 2x_i), \quad (1)$$

$$\dot{y}_i = x_i + a_i + D\xi_i(t), \quad (2)$$

where a_i is a parameter of the i th element. For a single FHN model, if $|a| > 1$, the system has only a stable fixed point, while $|a| < 1$ a limit cycle. The system with fixed point dynamics ($|a|$ slightly larger than one) is excitable because it will return to the fixed point only after a large excursion (“near limit cycle”) when perturbed away from the fixed point. To make the study more general, we suppose a_i is not the same for the elements, but has a uniform distribution $a_i \in (1, 1.1)$. This implementation of the model is physically significant because many physical systems are diffusively coupled, and nonidentity is more natural in physical situations. With nonidentical a_i , the uncoupled elements will have a different response to the same level of noise $\xi_i(t)$, e.g., different average firing frequency. The Gaussian noise is uncorrelated in different elements, i.e., $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$. A periodic boundary condition is employed in our studies. The parameter of the time scale is fixed at $\epsilon = 0.01$.

To characterize synchronization behavior in the lattice of nonidentical excitable systems, we introduce the phase of the elements [7]

$$\phi_i(t) = 2\pi \frac{t - \tau_k}{\tau_{k+1} - \tau_k} + 2\pi k, \quad (3)$$

where τ_k is the time of the k th firing of the element defined in simulations by threshold crossing of $x_i(t)$ at $x = 1.0$. The quantity

$$s_i = \sin^2\left(\frac{\phi_i - \phi_{i+1}}{2}\right) \quad (4)$$

measures the phase synchronization effect of neighboring elements [5]. A spatiotemporal average of s_i , i.e.,

$$S = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\frac{1}{N} \sum_{i=1}^N s_i \right) dt, \quad (5)$$

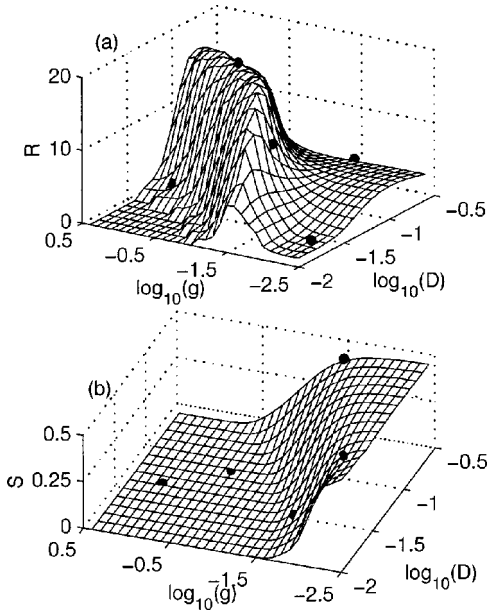


FIG. 1. Signal-to-noise ratio R (a) and degree of phase synchronization S (b) in the parameter space $(\log_{10}(g), \log_{10}(D))$ of a coupled lattice of nonidentical FHN neurons with $N=100$.

gives a measure of the degree of phase synchronization in the coupled system. For completely unsynchronized motion $S \approx 0.5$, while for globally synchronized system $S \approx 0$.

To measure the temporal coherence of the noise-induced motion, we examine the distribution of the pulse duration $T_k = \tau_{k+1} - \tau_k$. For a single element subject to noise, the distribution $P(T)$ has a peak at a certain value of T_k and an exponential tail at large values [10]. A measure of the sharpness of the distribution, for example,

$$R = \langle T_k \rangle / \sqrt{\text{Var}(T_k)}, \quad (6)$$

which can be viewed as signal-to-noise ratio, provides an indication of the coherence of the firing event. Biologically, this quantity is of importance because it is related to the timing precision of the information processing in neural systems [16]. For a single element, it has been shown that R possesses an optimal value at a certain level of noise [10]. Here we adopt the same measure of coherence, with the distribution $P(T)$ constructed by pulse duration of all the N elements during a long enough period of time.

Typical behaviors observed in numerical simulations of the system are in the following:

(i) Without external noise, each element comes to a fixed point state and no firing takes place. The fixed points are slightly different due to the nonidentity.

(ii) When strong enough noise is added to the originally quiescent system, the system is excited. The firing process of the elements may be in phase synchronization if the coupling is strong enough.

(iii) The coherence of the temporal and spatial pattern of the firing process is greatly enhanced in a region of noise and coupling strength.

The main features of the system are illustrated in Fig. 1, where the results of R and S in the parameter space $(\log_{10}(g), \log_{10}(D))$ are shown for a lattice of $N=100$ elements. For a fixed coupling g , R increases first with the noise

level D , reaches a maximum, then decreases again, showing the typical resonant behavior *without* an external signal. In general, for a stronger coupling, a higher level of noise is needed to excite the system. Similarly, for a fixed noise level D , R increases with increasing g until it reaches an optimal value; after that, it decreases again.

From Fig. 1, one can observe several dynamical regimes in the systems. For very weak coupling ($g < 10^{-2}$), the firing of the elements are essentially independent, because a noise-induced firing of an element cannot excite its neighboring lattices. Due to the independent firing, the phase difference has a uniform random distribution on $(0, 2\pi)$, resulting $S \approx 0.5$. In this region, with the increase of D , the temporal behavior of the system display coherence resonance similar to that observed in a single element. The enhancement of the coherence by the noise is not very pronounced. A maximal $R \approx 4.5$ is found around $D \approx 10^{-1}$. A typical spatiotemporal pattern of x_i and s_i for weakly coupled elements subject to relatively weak noise is shown in Fig. 2(a). Both the spatial and temporal behaviors are quite irregular. Each element has its individual average firing frequency due to the nonidentity (smaller frequency for larger a_i).

With the increase of the coupling strength, the system becomes sensitive to weak noise because the firing events induced by noise now become the source of excitation of the neighboring elements. This mutual excitation enhances the coherence of the motion in the coupled system, as indicated by increasing R and decreasing S . The lattice displays clusters of synchronization. The clusters break and reunite during the evolution, so that each element has slightly different firing frequency. Typical behavior of this partial synchronization regime is shown in Fig. 2(b). However, if the external noise is strong enough so that the noise dominates over the coupling, firing of each element is governed mainly by its individual noise, and synchronization clusters cannot survive, as seen in Fig. 2(c). The coherence of the temporal behavior is relatively low because quite large noise deforms greatly the ‘‘near limit’’ cycle. Note that firing frequency is not locked, but the difference is not as pronounced as in weak noise region [Fig. 2(a)]. The next regime where R takes large values ($R \sim 18$) while S is very close to zero, is the most interesting, because the system performs quite regular motion globally, as seen in Fig. 2(d). All elements are locked to a relatively large firing frequency, and the distribution of the pulse duration becomes very sharp. After that, with stronger coupling, the system keeps global synchronization, however, the temporal behavior becomes irregular again, as indicated by decreasing R . This can be understood qualitatively from the global dynamics $X = \langle x_i \rangle_N$ and $Y = \langle Y_i \rangle_N$, with $\langle \rangle_N$ denoting average over the lattice. From Eqs. (1) and (2), we get approximately

$$\epsilon \dot{X} = X - \frac{X^3}{3} - r^2 X - Y, \quad (7)$$

$$\dot{Y} = X + a_0 + \frac{D}{\sqrt{N}} \xi(t), \quad (8)$$

where $r^2 = \langle (x_i - X)^2 \rangle_N$ is the fluctuation level of local dynamics and higher order terms of this fluctuation are ignored;

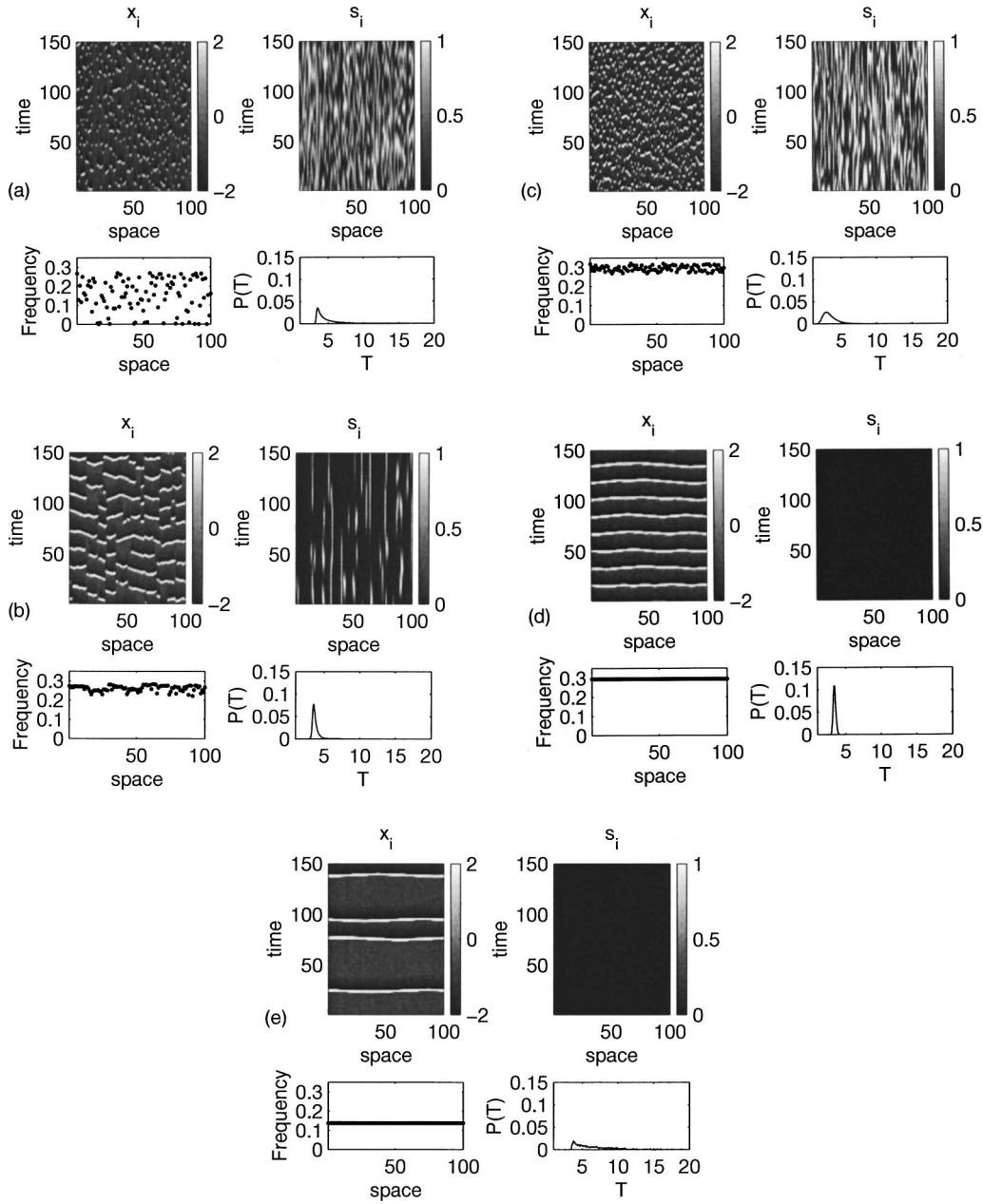


FIG. 2. Five typical dynamical regimes in the parameter space. The upper panels show the spatiotemporal structure of x_i and s_i . The time step is 0.2. The lower panels show the average firing frequency of the lattice and the distribution of the pulse duration. The location of these representative dynamical behaviors are shown by black dots in the parameters space in Fig. 1. The parameters are (a) $g = 0.005, D = 0.02$; (b) $g = 0.03, D = 0.025$; (c) $g = 0.02, D = 0.25$; (d) $g = 0.25, D = 0.07$; and (e) $g = 1.0, D = 0.04$.

$a_0 = \langle a_i \rangle_N$. The summation of the independent noise is still a Gaussian noise $\xi(t)$, but with a weaker strength D/\sqrt{N} . For strong enough coupling, the system achieves global synchronization, $x_i \approx X$, and $r^2 \approx 0$, and the coupled system can be viewed as a *single* element subject to a white noise with strength D/\sqrt{N} . For a single element, a rather weak noise may not excite the system, or the excited motion is quite irregular [10]. This explains the globally synchronized but irregular motion of the lattice, as shown in Fig. 2(e).

The locations of the above five representative dynamical regimes shown in Fig. 2 have been indicated by the black dots in Fig. 1. These regimes are typical for different sizes of the lattice. For a larger lattice, the regime of global regular motion (R large and $S \sim 0$) is wider in the parameter space.

Now let us discuss the phenomenon of *array enhanced coherence resonance*. For a fixed size of lattice and a certain value of coupling g , there is an optimal level of noise at which the system has a maximal value R_{max} . Figure 3 shows R_{max} as a function of g for different sizes of the lattice. The dashed line represents R_{max} for a single element with $a = 1.05$. As seen from this figure, in the weak coupling region, R_{max} of the coupled lattice is lower than that of a single element, because the nonidentity makes the distribution of the pulse duration broader. However, in some intermediate coupling region, even only *two* coupled elements can enhance the coherence of the temporal behavior, even though they are nonidentical. The enhancement is larger for larger N . For large enough lattice, R_{max} seems to be saturated, but

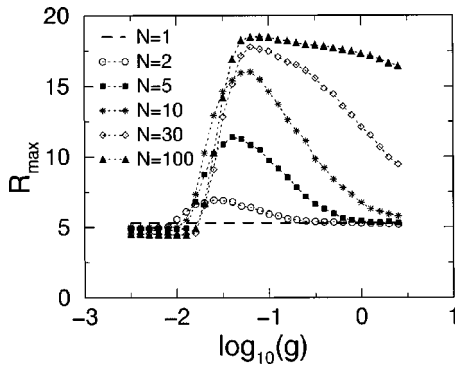


FIG. 3. Illustration of an array-enhanced coherence resonance. The maximal value of R is plotted as a function of g for various sizes of the lattice.

the region of coupling in which R_{max} takes large values is broader for larger lattice. As pointed above, for strong enough coupling, the coupled system tends to act as a single element, and R_{max} converges to that of the single element at large value of g . Clearly, for a larger lattice, stronger coupling is needed to make the whole lattice act as a single element due to the local diffusive coupling, resulting in a slower convergence of R_{max} to that of the single element.

The properties observed here have some similarity to that in coupled bistable stochastic resonance oscillators *subject to*

the same periodic signal [8]. The difference is that in Refs. [8] and [14], global synchronization is observed only at an optimal level of noise, while in the present system, global synchronization occurs for strong enough coupling if the noise level is not too high. The global motion can be regular or irregular. Here we should emphasize that there is no periodic signal in our system, and the coherent motion is *purely induced by noise and enhanced by coupling*.

To conclude, we demonstrate global synchronization and array enhanced coherence in a lattice of locally coupled, non-identical FHN model neurons and the nonidentity demonstrates robustness of the phenomena. The results are similar for identical neurons. Interaction between the elements not only renders synchronization of the firing process induced by independent noise, but also improves the coherence of the noise-induced motion greatly. The phenomena demonstrated may be of importance in neurophysiology, where background noise may play an active role to increase the order and timing precision of a large ensemble of interacting neurons in biological information processing.

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